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Hidden supersymmetries in supersymmetric quantum mechanics

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Abstract

We discuss the appearance of additional, hidden supersymmetries for simple $0+1$ $Ad(G)$-invariant supersymmetric models and analyse some geometrical mechanisms that lead to them. It is shown that their existence depends crucially on the availability of odd order invariant skewsymmetric tensors on the (generic) compact Lie algebra $\mathcal{G}$, and hence on the cohomology properties of the Lie algebra considered.

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1 Introduction

In supersymmetric quantum mechanics models with standard supersymmetry, the supercharges Q_a are related to the Hamiltonian H via $\{Q_a, Q_b\} = H\delta_{ab}$, $a, b = 1, \ldots, N$. In many of these models one can find additional or 'hidden' supercharges $\tilde{Q}$ [1, 2], involving the structure constants of a Lie algebra, and perhaps a Killing-Yano tensor [3, 4]. The appearance of the Killing-Yano tensor in this context is not surprising, since it also plays a role in the existence of hidden symmetries [5, 6].

The additional supercharges are required to satisfy

$$\{Q_a, \tilde{Q}\} = 0 \quad ; \tag{1}$$

hence $[\tilde{Q}, H] = 0$, so that the $\tilde{Q}$'s generate supersymmetries of the theory. We shall consider three models: one with bosonic superfields, and two with fermionic superfields (with $N = 1$ and $N = 2$ respectively) for which the bosonic component variables are auxiliary [7].

A typical example of the bosonic superfield case, in which the Lie algebra $\mathcal{G}$ is $su(2)$, is that of the non-relativistic motion of a spin-$\frac{1}{2}$ particle in the background field of a Dirac monopole [8] (see also [9]). The case $G = SU(2)$ is, however, rather exceptional since it is the only group for which the structure constants of its algebra coincide with the fully antisymmetric tensor (ϵ_{ijk}) of a ($\dim G$)-dimensional space. Thus, a natural question to ask is what generalisations are possible when simple (and compact) algebras $\mathcal{G}$ of rank $l > 1$ are employed. Also, for $l > 1$ there exist other available skewsymmetric tensors (of odd order > 3): they are provided by the higher order cocycles of the Lie algebra cohomology of $\mathcal{G}$. Thus, in order to investigate the appearance of hidden charges in a group theoretical context, we look first at

$$\tilde{Q}_3 = \dot{x}_i f_{ij} \psi_j - \tfrac{1}{3} i f_{ijk} \psi_i \psi_j \psi_k \quad , \tag{2}$$

where x_i and ψ_i are the position and fermionic coordinates respectively, $i = 1, \ldots, \dim \mathcal{G}$, and f_{ij} is an antisymmetric second order Killing-Yano tensor [3, 4] associated with the structure constants f_{ijk} in such a way that (1) holds, and second we look at

$$\tilde{Q}_5 = \dot{x}_i f_{ijkl} \psi_j \psi_k \psi_l - \tfrac{1}{5} i \Omega_{ijkpq} \psi_i \psi_j \psi_k \psi_p \psi_q \tag{3}$$

in various contexts. In (3), $f_{ijkl} = f_{[ijkl]}$ is a fourth order generalised Killing-Yano tensor and Ω_{ijkpq} is a fifth order totally antisymmetric invariant tensor, associated with the third order (Racah-Casimir) invariant symmetric polynomial of such $\mathcal{G}$ as allow for one. This exists for $su(n)$, $n \geq 3$, and $su(3)$ will be good enough to illustrate the extent of most of our results when using the fifth order cocycle.

In the fermionic superfield case, it is possible to construct models where the only dynamical fields are fermionic, and whose Lagrangian includes an interaction term constructed using the structure constants of the simple Lie algebra. Our aim is to construct, in terms of the corresponding fermionic variables and the higher order cocycles, hidden supercharges in the $N = 1$ and $N = 2$ cases. When $N = 1$ we shall restrict the Lie algebra to $su(n)$, whereas in the $N = 2$ case the simple Lie algebra $\mathcal{G}$ will be unrestricted. The restriction reflects the fact that the discussion of the $N = 1$ case employs the identity $C_{ijk}\Omega_{ijks_4\ldots s_{2m-1}} = 0$, which we believe holds for all $\mathcal{G}$, but for which explicit detailed proofs are available [10] only for $su(n)$.

We consider only $Ad(G)$-invariant simple $0 + 1$ supersymmetry models. In Sec. 2 we describe the case of a non-relativistic system moving in a space that is the representation space of the adjoint representation of the symmetry group G, coupled to a background potential A_i. It is shown in sec. 3 that in the free case there exist non-standard supersymmetries associated with all the higher order cocycles. In the presence of the background field A_i (sec. 4), however, we find only one hidden supercharge for the lowest order cocycle *i.e.*, for that given by the structure constants of $\mathcal{G}$.

In Sec. 5, we consider a $N = 1$ simple purely fermionic model and show that one may also construct hidden supercharges from each of the l $su(n)$ algebra cohomology cocycles. In Sec. 6 the case with $N = 2$ is considered. It is shown that we may construct two hidden supercharges for each of the l cocycles of the Lie algebra $\mathcal{G}$.

2 A particle model with bosonic and fermionic degrees of freedom

Let G be a compact, simple Lie group of algebra $\mathcal{G}$, $[X_i, X_j] = if_{ijk}X_k$. We set out from the superspace Lagrangian

$$\mathcal{L} = \tfrac{1}{2}i\dot{\Phi}_i D\Phi_i + iqD\Phi_i A_i(\Phi) = K + \theta L \ , \quad i = 1, \ldots, \dim \mathcal{G} \ , \qquad (4)$$

where θ is a real Grassmann variable, the $\Phi_i = \Phi_i(t,\theta)$ are scalar superfields, and the covariant derivative D and the generator of supersymmetry Q are given by

$$\Phi_i(t,\theta) = x_i(t) + i\theta\psi_i(t) \ , \quad D = \partial_\theta - i\theta\partial_t \ , \quad Q = \partial_\theta + i\theta\partial_t \ . \tag{5}$$

The Lagrangian is invariant under the (real, adjoint) action of G. Using

$$\begin{aligned} D\Phi_i &= i(\psi_i - \theta\dot{x}_i) \ , \\ A_i(\Phi) &= A_i(x) + i\theta\psi_j\partial_j A_i(x) \ , \end{aligned} \tag{6}$$

the expansion (4) of $\mathcal{L}$ gives

$$K = -\tfrac{1}{2}\dot{x}_i\psi_i - q\psi_i A_i \ , \tag{7}$$

$$L = \tfrac{1}{2}\dot{x}_i\dot{x}_i + \tfrac{1}{2}i\psi_i\dot{\psi}_i + q\dot{x}_i A_i - \tfrac{1}{2}iqF_{ij}\psi_i\psi_j \ , \tag{8}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i \ . \tag{9}$$

We easily find the momenta and canonical commutators

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i + qA_i \ , \quad [x_i, p_j] = i\delta_{ij} \ , \quad \{\psi_i, \psi_j\} = \delta_{ij} \ , \tag{10}$$

and compute

$$H = \tfrac{1}{2}\dot{x}_i\dot{x}_i + \tfrac{1}{2}iqF_{ij}\psi_i\psi_j \ . \tag{11}$$

The standard supersymmetry that leaves the action for (4) invariant is $\delta\Phi_i = -i\epsilon Q\Phi_i$, *i.e.*

$$\delta x_i = -i\epsilon\psi_i \ , \quad \delta\psi_i = \epsilon\dot{x}_i \ . \tag{12}$$

Noether's theorem

$$-i\epsilon Q = \delta x_i p_i + \delta\psi_i \frac{\partial L}{\partial\dot{\psi}_i} - i\epsilon K \ , \tag{13}$$

where the piece depending on K comes from the quasi-invariance of the Lagrangian, gives for the conserved supercharge

$$Q = \dot{x}_i\psi_i \quad . \tag{14}$$

It is easy to check that Q above generates (12) by means of the canonical formalism, and that

$$Q^2 = H \tag{15}$$

reproduces the right hand side of (11).

3 Hidden supersymmetries in the free case

3.1 The case of $\tilde{Q}_3$

Here we shall put $q = 0$ in expressions in Sec. 2. We intend first to seek an additional supersymmetry $\tilde{Q}_3$ such that $\{Q, \tilde{Q}_3\} = 0$ in the form

$$\tilde{Q}_3 = \dot{x}_i f_{ij} \psi_j - \tfrac{1}{3} i f_{ijk} \psi_i \psi_j \psi_k \quad , \tag{16}$$

where f_{ij} is to be determined. Since it is easier to work classically, we use the Dirac bracket formalism corresponding to (10), so that for F and G functions of dynamical variables

$$\{F, G\} = \frac{\partial F}{\partial x_l}\frac{\partial G}{\partial p_l} - \frac{\partial F}{\partial p_l}\frac{\partial G}{\partial x_l} + i(-1)^F \frac{\partial F}{\partial \psi_l}\frac{\partial G}{\partial \psi_l} \ . \tag{17}$$

Using $Q = p_i \psi_i$ from (14) and (16), we get

$$\{Q, \tilde{Q}_3\} = -\psi_l \dot{x}_i f_{ij,l} \psi_j - i \dot{x}_l (\dot{x}_i f_{il} - i f_{ljk} \psi_j \psi_k) \ , \tag{18}$$

where $f_{ij,l} = \frac{\partial}{\partial x^l} f_{ij}$. The second term in the r.h.s. of (18) is zero if

$$f_{ij} = -f_{ji} \ , \tag{19}$$

and the other two cancel if

$$f_{i[j,l]} = f_{ijl} \ . \tag{20}$$

We could even have written $f_{ij,k}$ instead of f_{ijk} in (16), and then $f_{ij,k}$ may effectively be replaced by $f_{[ij,k]}$ in virtue of the $\psi_i \psi_j \psi_k$ factor. Then (20) is

$$f_{i[j,l]} = f_{[ij,l]} \quad , \tag{21}$$

so that

$$\partial_{(l} f_{i)j} = 0 \tag{22}$$

or

$$2 f_{ij,l} = f_{li,j} + f_{jl,i} \ . \tag{23}$$

Equations (21) or (22) state that the derivative of the antisymmetric tensor (19) is also skewsymmetric, and characterise f_{ij} as Killing-Yano tensor [3, 4]. One way to satisfy this condition sets

$$f_{ij} = f_{ijk} x_k \ , \tag{24}$$

giving

$$\tilde{Q}_3 = L_i\psi_i + \tfrac{2}{3}S_i\psi_i \quad , \tag{25}$$

where

$$L_i = f_{ijk}x_j\dot{x}_k \quad , \quad S_i = -\tfrac{1}{2}if_{ijk}\psi_j\psi_k \quad . \tag{26}$$

Furthermore,

$$\begin{aligned} [L_i, x_j] &= if_{ijk}x_k \ , \quad [S_i, \psi_j] = if_{ijk}\psi_k \ , \\ [\tilde{Q}_3, x_i] &= if_{ijk}x_j\psi_k \ , \quad \{\tilde{Q}_3, \psi_i\} = L_i + 2S_i \ . \end{aligned} \tag{27}$$

Both L_i and S_i are representations of the Lie algebra $\mathcal{G}$ since they obey

$$[L_i, L_j] = if_{ijk}L_k \quad , \quad [S_i, S_j] = if_{ijk}S_k \ . \tag{28}$$

We note that the $\tilde{Q}_3$ supersymmetry does not close on the Hamiltonian, but instead we have

$$\{\tilde{Q}_3, \tilde{Q}_3\} = \vec{J}^2 + \tfrac{1}{3}\vec{S}^2 \quad , \quad J_i = L_i + S_i \quad , \tag{29}$$

where J_i is the conserved charge associated with the G-invariance of the action (4). The result here parallels the result of [8] for particle motion in the background field of a Dirac monopole.

We note in passing also that $\tilde{Q}_3$ looks similar to the Kostant fermionic operator K [11, 12]

$$K = \rho_i\gamma^i - \frac{i}{3!2}f_{ijk}\gamma^i\gamma^j\gamma^k \ , \tag{30}$$

where ρ refers to some representation of $\mathcal{G}$ (*cf* L_i in (26)). In (30) the quantised fermion operators ψ_i have been represented, $\psi_i \mapsto \frac{1}{\sqrt{2}}\gamma_i$, by the Dirac matrices of a euclidean space of dimension $\dim\mathcal{G}$ which obey $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. However, (30) implies $K^2 = \vec{\rho}^2 + \frac{1}{3}\vec{S}^2$. Clearly K is not proportional the supercharge $\tilde{Q}_3$ of (16), for any representation ρ of $\mathcal{G}$.

3.2 Other hidden supersymmetries

We generalise now the previous paragraph to supercharges involving the higher order cocycles of the Lie algebra $\mathcal{G}$. Instead of (16), we consider $\tilde{Q}_5$ as in (3)

$$\tilde{Q}_5 = p_i f_{ijkp}\psi_{jkp} - \tfrac{1}{5}i\Omega_{ijkpq}\psi_{ijkpq} \quad , \tag{31}$$

where $f_{ijkp} = f_{ijkp}(x)$,

$$f_{ijkp} = f_{i[jkp]} \ , \tag{32}$$

we have used the abbreviation $\psi_{ijk\ldots} = \psi_i \psi_j \psi_k \ldots$, and Ω_{ijkpq} is by definition totally antisymmetric in all its five indices. Instead of (32) we might consider the replacement of Ω_{ijkpq} by

$$f_{[ijkp,q]} \ , \tag{33}$$

where $f_{ijkp,q} = \partial f_{ijkp}/\partial x_q$. We now demand that $\tilde{Q}_5$ anticommutes with $Q = p_i \psi_i$

$$\{Q, \tilde{Q}_5\} = 0 \tag{34}$$

quantum mechanically for variety, and also because promotion of the classical calculation in this context is not in this instance trivial. Thus

$$\begin{aligned} \{Q, \tilde{Q}_5\} &= p_l \{\psi_l, \tilde{Q}_5\} + [\tilde{Q}_5, p_l]\psi_l \\ &= p_l (3 p_i f_{ijkl} \psi_{jk} - i \Omega_{ljkpq} \psi_{jkpq}) + i p_l f_{ljkp,q} \psi_{jkpq} \ . \end{aligned} \tag{35}$$

The first term of (35) can be eliminated by requiring that f_{ijkl} is antisymmetric in i and l, and, hence, using (33) that

$$f_{ijkp} = f_{[ijkp]} \ . \tag{36}$$

The other result needed to secure (34) is the vanishing of the part of

$$f_{ljkp,q} - \Omega_{ljkpq} \tag{37}$$

antisymmetric in $jkpq$. If we try the ansatz

$$f_{ljkp} = \Omega_{ljkpq} x_q \quad , \tag{38}$$

then this requirement is satisfied. Then

$$\tilde{Q}_5 = p_i \Omega_{ijkpq} x_q \psi_{jkp} - \tfrac{1}{5} i \Omega_{ijkpq} \Psi_{ijkpq} \tag{39}$$

is a hidden, AdG-invariant supercharge. This situation is one that applies to $\tilde{Q}_7$, involving Ω_7, etc. Actually, since the charges are constructed using the structure constants and the higher-order cocycles, there will be a hidden supercharge for each cocycle in a model with an arbitrary simple compact group. We may then conclude the following:

The $Ad(G)$-invariant free supersymmetric particle model described by $\mathcal{L} = \frac{1}{2}\Phi_i D \Phi_i$ admits l hidden supercharges $\tilde{Q}_s$, $s = 1, \ldots, l$. These are determined

by the l Lie algebra cohomology cocycles of the simple compact algebra $\mathcal{G}$ of rank l.

If we had written $f_{[ijkp,q]}$ in the second term of (31), then (37) corresponds to the vanishing of the part of

$$f_{l[jkp],q} = f_{[ljkp,q]} \tag{40}$$

antisymmetric in $jkpq$. To extract the minimal condition, we write

$$f_{[ljkp,q]} = \tfrac{4}{5} f_{l[jkp,q]} + \tfrac{1}{5} f_{[jkpq],l} \ . \tag{41}$$

Then (40) is equivalent to

$$f_{[ljkp,q]} = f_{[jkpq],l} \quad , \tag{42}$$

which is an analogue of (23) and, together with the complete antisymmetry of f_{ijkl}, means that f_{ijkl} is a Killing-Yano tensor of valence four. This is consistent with Tanimoto's analysis [6], although in a slightly different context. The previous result may be rephrased in the following form:

The additional supersymmetry exists because each Lie algebra cocycle of order $2m-1$ provides a Killing-Yano tensor of valence $2(m-1)$ by the analogous to (38) since then

$$f_{ii_1 \ldots i_{2m-3},q} + f_{qi_1 \ldots i_{2m-3},i} = 0 \quad . \tag{43}$$

One might look for different solutions, but we did not find any. For example, the condition (42) defeats the otherwise interesting ansatz $f_{ijkl} = f_{[ij} f_{kl]}$, where f_{ij} is a second order Killing-Yano tensor. In fact, the Killing-Yano condition appears to be very restrictive, and it is likely that (24) and (38) are the only possible solutions for the cases considered.

4 $\tilde{Q}_3$ in the presence of a background field

We investigate now to what extent it is possible to reproduce the analysis of sec. 3 in the $A \neq 0$ case. When Q is given by (14) and $\tilde{Q}_3$ by (16), where now $p_i = \dot{x}_i + qA_i$ as in Sec. 2, we find classically that $\{Q, \tilde{Q}_3\} = 0$ requires

$$\begin{aligned} \{Q, \tilde{Q}_3\} \quad = \quad & -\psi_l \dot{x}_i f_{ij,l} \psi_j - i\dot{x}_l (\dot{x}_i f_{il} - i f_{ljk} \psi_j \psi_k) \\ & + q f_{lj} F_{il} \psi_i \psi_j = 0 \quad , \end{aligned} \tag{44}$$

so we have to impose (19), (23) and

$$f_{l[j}F_{i]l} = 0 \tag{45}$$

where F_{ij} is given by (9). It is sensible to try first the solution

$$f_{ij} = f_{ijk}x_k \tag{46}$$

of (19) and (23). To satisfy (45) we may choose

$$F_{ij} = x_i y_j - x_j y_i \quad , \tag{47}$$

where $y_i = d_{ijk}x_j x_k$, whenever there exists an invariant symmetric tensor d_{ijk} on the algebra (which excludes $su(2)$). This is true because of the identities $f_{ijk}x_j x_k = 0 = f_{ijk}x_j y_k$, the last one due to the invariance of the symmetric tensor d_{ijk}. However consistency with (9) demands that any ansatz for F_{ij}, and (47) in particular, obeys

$$\partial_{[i}F_{jk]} = 0 \quad . \tag{48}$$

The simple ansatz $F_{ij} = a(Z)f_{ijk}x_k$ where $Z = x_k x_k$, satisfies (48) when the algebra is $su(2)$, in which case $a(Z) \propto Z^{-3/2}$, as is expected for the monopole [8]. But for $su(n)$, *e.g.* for $su(3)$, for which $Z = x_k x_k$ and $Y = d_{ijk}x_i x_j x_k = x_k y_k$, the ansatz $F_{ij} = a(Z,Y)f_{ijk}x_k$ fails. We can see this by contracting

$$0 = \partial_{[i}F_{jk]} = 2\frac{\partial\, a}{\partial\, Z}x_{[i}f_{jk]l}x_l + 3\frac{\partial\, a}{\partial Y}y_{[i}f_{jk]l}x_l + a\, f_{ijk}\ , \tag{49}$$

with y_i, which gives

$$\tfrac{2}{3}Y\,\frac{\partial\, a}{\partial\, Z}f_{jkl}x_l + y^2\frac{\partial\, a}{\partial\, Y}f_{jkl}x_l + a\, f_{jkl}y_l = 0 \quad . \tag{50}$$

The independence of the tensors $f_{ijk}x_l$ and $f_{jkl}y_l$ now gives $a = 0$. The method used here can be extended to show the failure also of the more general ansatz

$$F_{ij} = a(Z,Y)f_{ijk}x_k + b(Z,Y)f_{ijk}y_k\ . \tag{51}$$

The case of $su(2)$ of course escapes such a failure because y_i then does not exist.

The choice (47) can be used for $su(n)$, $n \geq 3$: it obeys (48) directly but also allows us to write down suitable choices of A_i for which (9) reproduces (47). For example, we may use

$$A_i = -\tfrac{1}{3} Y x_i \quad . \tag{52}$$

More generally, any A_i of the form

$$A_i = \alpha(Z,Y)x_i + \beta(Z,Y)y_i \tag{53}$$

gives F_{ij} in suitable form:

$$F_{ij} = \left(2\frac{\partial \beta}{\partial Z} - 3\frac{\partial \alpha}{\partial Y} \right)(x_i y_j - x_j y_i) \ . \tag{54}$$

Thus, *for any G-invariant model (4) there exists an additional supersymmetry $\tilde{Q}_3$ given by*

$$\tilde{Q}_3 = p_i f_{ijk}\psi_j x_k - \tfrac{1}{3} i f_{ijk}\psi_i\psi_j\psi_k \ , \tag{55}$$

determined by the structure constants f_{ijk} of $\mathcal{G}$. The contribution proportional to A_i disappears from the first term of (54) again because of the identities $f_{ijk}x_j x_k = 0 = f_{ijk}x_j y_k$. But this does not mean that we have recovered the free case, because A_i is present in $Q = (p_i - qA_i)\psi_i$. $\tilde{Q}$ satisfies the relations

$$[\tilde{Q}_3, x_i] = i f_{ijk} x_j \psi_k \ , \ \{\tilde{Q}_3, \psi_i\} = L_i + 2S_i \ , \ \{\tilde{Q}_3, \tilde{Q}_3\} = (\vec{L}+\vec{S})^2 + \tfrac{1}{3}\vec{S}^2 \ , \tag{56}$$

where now $L_i = f_{ijk}x_j p_k$.

We have not found, however, an analogue of $\tilde{Q}_5$ for $q \neq 0$. If the Killing-Yano tensor is $f_{ijkl} = \Omega_{ijklm}x_m$ and F_{ij} is proportional to (47), the condition corresponding to (45),

$$f_{l[jmn}F_{i]l} = 0 \ , \tag{57}$$

is not satisfied because, in contrast with $f_{ijk}x_j y_k = 0$, $\Omega_{ijklm}x_l y_m \neq 0$. It is possible that the use of more general background fiels (see [13]) opens the way to richer possibilities.

5 $N = 1$ Fermion superfields

5.1 Basic formalism

We turn now to a different supersymmetric model in which hidden supersymmetries related to higher order cocycles also occur. This model is a theory of fermions with all states in one energy level, and without bosonic dynamical variables [7]. Consider the $Ad(SU(n))$-invariant superspace Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\Lambda_i D\Lambda_i + \frac{1}{3!} i g f_{ijk}\Lambda_i\Lambda_j\Lambda_k \quad , \tag{58}$$

where the $\Lambda_i = \Lambda_i(t,\theta)$ are $i = 1,\ldots,\dim\mathcal{G}$ fermionic superfields,

$$\Lambda_i(t,\theta) = \psi_i(t) + \theta B_i(t) \ , \quad D\Lambda_i = B_i - i\theta\dot{\psi}_i \ . \tag{59}$$

The expansion of (58) may be written as $\mathcal{L} = K + \theta L$, where now

$$\begin{aligned} K &= \tfrac{1}{2}\psi_i B_i + \tfrac{1}{3!} i g \psi_i\psi_j\psi_k \quad , \\ L &= \tfrac{1}{2}(i\dot{\psi}_i\psi_i + B_i B_i + i g f_{ijk}\psi_i\psi_j B_k) \ . \end{aligned} \tag{60}$$

As in Sec. 2, we may use the expression of K and L in Noether's theorem (see (13)) for the variations

$$\delta\psi_i = -\epsilon B_i \ , \quad \delta B_i = i\epsilon\dot{\psi}_i \quad , \tag{61}$$

to obtain

$$Q = \frac{1}{6} i g f_{ijk}\psi_i\psi_j\psi_k \ . \tag{62}$$

All the bosonic components B_i are auxiliary. Using their Euler-Lagrange equations to solve for B_i, one obtains the classical Lagrangian $L_c = \frac{1}{2} i\psi_i\dot{\psi}_i$. The classical Hamiltonian vanishes identically [2], but the quantum Hamiltonian, defined by $H = Q^2$, and computed using $\{\psi_i, \psi_j\} = \delta_{ij}$, is not zero but a constant,

$$Q^2 = \frac{1}{48}\, g^2\, n(n^2 - 1) \tag{63}$$

for $su(n)$.

5.2 Hidden supersymmetries in the fermionic model

As for the (4) model for $q = 0$, there exist in this case additional supercharges $\tilde{Q}$ for every non-trivial cocycle of any $su(n)$ Lie algebra. To see this, let

[2] There is an $S_i S_i$ part, which is proportional to $f_{mij} f_{mkl}\psi_i\psi_j\psi_k\psi_l$ and which is zero classically by virtue of the Jacobi identity and the fermionic character of the ψ's.

$\Omega^{(2m-1)}_{i_1\ldots i_{2m-1}}$ be a non-trivial cocycle corresponding to an invariant symmetric tensor t of order m. If t has components $t_{l_1\ldots l_m}$, then the $(2m-1)$-order cocycle is given by[3]

$$\Omega^{(2m-1)}_{i_1\ldots i_{2m-1}} = f^{l_i}_{[i_1 i_2} \ldots f^{l_{m-1}}_{i_{2m-3} i_{2m-2}} t_{i_{2m-1}] l_i \ldots l_{m-1}} \ . \tag{64}$$

Using it, we form

$$\tilde{Q}_{2m-1} = \Omega^{(2m-1)}_{i_1\ldots i_{2m-1}} \psi_{i_1\ldots i_{2m-1}} \ , \tag{65}$$

where

$$\psi_{i_1\ldots i_{2m-1}} \equiv \psi_{i_1} \ldots \psi_{i_{2m-1}} \ . \tag{66}$$

We compute the quantum anticommutator of Q in (62) and $\tilde{Q}_{2m-1}$ by expressing the products $\psi_{i_i i_2 i_3} \psi_{j_1\ldots j_{2m-1}}$ and $\psi_{j_1\ldots j_{2m-1}} \psi_{i_i i_2 i_3}$ as linear combinations of completely antisymmetrised products of ψ's. This can be done by repeated use of the identities (deduced from $\{\psi_i, \psi_j\} = \delta_{ij}$)

$$\begin{aligned} \psi_i \psi_{j_1\ldots j_p} &= \psi_{i j_1 \ldots j_p} + \tfrac{1}{2} p \delta_{[i j_1} \psi_{j_2 \ldots j_p]} \ , \\ \psi_{j_1\ldots j_p} \psi_i &= \psi_{j_1 \ldots j_p i} + \tfrac{1}{2} p \psi_{[j_1 \ldots j_{p-1}} \delta_{j_p] i} \ . \end{aligned} \tag{67}$$

Then we easily find that the terms with no δ's or an even number of them vanish identically because the two contributions coming from the anticommutator cancel each other. So we are left with

$$\begin{aligned} \{Q, \tilde{Q}_{2m-1}\} &= \tfrac{1}{6} i g [3(2m-1) f_{k i_2 i_3} \Omega^{(2m-1)}_{k j_2 \ldots j_{2m-1}} \psi_{i_2 i_3 j_2 \ldots j_{2m-1}} \\ &\quad - \tfrac{1}{4} \frac{(2m-1)!}{(2m-4)!} f_{klm} \Omega^{(2m-1)}_{klm j_4 \ldots j_{2m-1}} \psi_{j_4 \ldots j_{2m-1}}] \ . \end{aligned} \tag{68}$$

The first term in (68) vanishes due to the Jacobi identity (since the indices $i_2 i_3 j_2 \ldots j_{2m-1}$ are antisymmetrised due to the presence of the ψ's) and the second also vanishes because the maximal contraction of indices among the above two $su(n)$ cocycles of different order gives zero [10]. Hence, *the* l $\tilde{Q}_{2m-1}$ *define new conserved fermionic charges of higher order.* As in the case of Q in (62), they square to a constant. For example, let us consider the case of $\tilde{Q}_5$ for $su(n)$, $n \geq 3$. The square of $\tilde{Q}_5$ is given by

$$\tilde{Q}_5^2 = \Omega^5_{i_1\ldots i_5} \Omega^5_{j_1\ldots j_5} \psi_{i_1\ldots i_5} \psi_{j_1 \ldots j_5} \ . \tag{69}$$

It is shown in [16] (see also [10]) that this square is a number. Hence, Q, $\tilde{Q}_5$ and $\mathbf{I}$ close into a superalgebra.

[3]For the relation among symmetric tensors and cocycles of generic compact simple Lie algebra $\mathcal{G}$ see [14], [10] and [15].

6 $N=2$ Fermion superfields

6.1 Basic formalism

We now consider a purely fermionic model with two standard supersymmetries [7] (see also [17]). The supersymmetry algebra in terms of the covariant derivatives for this model is

$$D = \partial_\theta - i\theta^* \partial_t \ , \quad D^* = -\partial_{\theta^*} + i\theta\partial_t \ , \quad \{D, D^*\} = 2i\partial_t \ . \tag{70}$$

The $N=1$ superfields Λ_i are replaced by $N=2$ superfields $\Psi_i = \Psi_i(t,\theta,\theta^*)$, $i=1,\ldots,\dim\mathcal{G}$, to which the chirality condition $D^*\Psi_i = 0$ is imposed. This of course means that Ψ_i^* obeys $D\Psi_i^* = 0$ and is antichiral. Solving as usual the chirality condition we obtain the superfield expansions

$$\begin{aligned}
\Psi_i &= e^{i\theta^*\theta\partial_t}(\mu_i - \theta B_i) = \mu_i - \theta B_i + i\theta^*\theta\dot{\mu}_i \\
\Psi_i^* &= \mu_i^* - \theta^* B_i^* - i\theta^*\theta\dot{\mu}_i^* \ ,
\end{aligned} \tag{71}$$

where μ_i are fermionic and B_i are bosonic. The following $Ad(G)$-invariant superspace action has the property that the B's are non-dynamical, and includes an interaction term:

$$\begin{aligned}
S &= -\tfrac{1}{2}\int dt d\theta d\theta^* \Psi_i \Psi_i^* \\
&\quad +\tfrac{1}{6}\int dt d\theta i C_{ijk}\Psi_i\Psi_j\Psi_k + \tfrac{1}{6}\int dt d\theta^* i C_{ijk}\Psi_i^*\Psi_j^*\Psi_k^* \ ,
\end{aligned} \tag{72}$$

where C_{ijk} are the structure constants of the Lie algebra $\mathcal{G}$. The $Ad(G)$-invariant component Lagrangian is given by

$$L = \tfrac{1}{2}i(\mu_i^*\dot{\mu}_i + \mu_i\dot{\mu}_i^*) + \tfrac{1}{2}B_i^*B_i - \tfrac{1}{2}iC_{ijk}\mu_i\mu_jB_k - \tfrac{1}{2}iC_{ijk}\mu_i^*\mu_j^*B_k^* \ . \tag{73}$$

The Euler Lagrange equations of the B_i, B_i^* are algebraic and can be used to eliminate these variables. Then, by writing $J_i = -\frac{1}{2}iC_{ijk}\mu_j\mu_k$, the Lagrangian becomes

$$L = \tfrac{1}{2}i(\mu_i^*\dot{\mu}_i + \mu_i\dot{\mu}_i^*) - 2J_iJ_i^* \ . \tag{74}$$

The canonical formalism yields the Dirac brackets

$$\{\mu_i, \mu_j^*\} = -i\delta_{ij}, \quad \{\mu_i, \mu_j\} = 0, \quad \{\mu_i^*, \mu_j^*\} = 0 \ , \tag{75}$$

and classically we have

$$H = 2J_iJ_i^* \ . \tag{76}$$

The non-zero supersymmetry variations of the fields μ_i and B_i are given by

$$\begin{aligned} \delta_{\epsilon^*} B_i &= -2i\epsilon^* \dot{\mu}_i \ , \quad \delta_{\epsilon^*} \mu_i^* = \epsilon^* B_i^* \ , \\ \delta_{\epsilon} B_i^* &= -2i\epsilon \dot{\mu}_i^* \ , \quad \delta_{\epsilon} \mu_i = \epsilon B_i \ . \end{aligned} \tag{77}$$

These variations correspond, via Noether's theorem, to the conserved charges

$$Q = \tfrac{1}{3} i C_{ijk} \mu_i \mu_j \mu_k \ , \quad Q^* = \tfrac{1}{3} i C_{ijk} \mu_i^* \mu_j^* \mu_k^* \ . \tag{78}$$

When we quantise the theory we shall regard the μ_i as the creation operators. Hence, to avoid confusion, the following replacements will be made from now on: $\mu_i^* = \pi_i$, $\mu_i = c_i$, as in [17], so that $\pi^* = c_i$. Thus in the quantum theory, we have the anticommutation relations

$$\{c_i \ , \ \pi_j\} = \delta_{ij} \ . \tag{79}$$

Also the quantum mechanical Hamiltonian is defined via

$$\{Q \ , \ Q^*\} = 2H_q \ , \tag{80}$$

which gives the result

$$H_q = \{J_i \ , \ J_i^*\} - \tfrac{1}{6} c^2 \ , \tag{81}$$

where $c^2 = C_{ijk} C_{ijk}$ ($= n(n^2 - 1)$ for $su(n)$), and where it should be noted that J_i and J_i^* do not commute. One might expect that that H_q is closely related to the quadratic Casimir operator $X^2 = X_i X_i$ of $\mathcal{G}$, where $X_i = -i C_{ijk} \, c_j \, \pi_k$. It is simple to confirm this for the case of $su(n)$ by proving the following identities

$$\begin{aligned} X_i X_i &= nN - 2J_i J_i^* \\ &= n(n^2 - 1 - N) - 2J_i^* J_i \quad , \end{aligned} \tag{82}$$

where $N = c_i \pi_i$ is the total fermion number operator, and $\pi_i c_i = (n^2 - 1 - N)$. These allow the commutator and anticommutator of J_i and J_i^* to be calculated, and give rise to the result

$$H_q = \tfrac{1}{3} \, n(n^2 - 1) - X_i X_i \ . \tag{83}$$

The results (81) and (83), viewed together, seem a strangely related pair. However their agreement, as well as the correctness of (82), can easily be confirmed by considering actions of the operators in question on each of the

fermion number $N = 0,\ 1,\ 2,\ 3$ states of the simple but non-trivial $SU(2)$ version of the theory. Further, having set out from the definition (80) of H_q, we know that all energy eigenvalues are non-negative.

In addition to

$$q_{30} = Q/2 = \tfrac{1}{3!} i C_{ijk} c_i c_j c_k \quad , \quad q_{03} = q_{30}^* = Q^*/2 \ , \tag{84}$$

in which the first and second subscripts indicate the numbers of c_i and π_i factors respectively, two further fermionic operators occur naturally:

$$q_{21} = \tfrac{1}{2}\, i C_{ijk} c_i c_j \pi_k \ , \quad q_{12} = \tfrac{1}{2}\, i C_{ijk} c_i \pi_j \pi_k \ . \tag{85}$$

These operators each anticommute with each of q_{30} and q_{03} and obey

$$\{q_{21}\ ,\ q_{12}\} = \tfrac{1}{2} X_i X_i \quad , \quad q_{21}^2 = 0 \quad , \quad q_{12}^2 = 0 \ . \tag{86}$$

It follows that q_{21} and q_{12} commute with $X_i X_i$ and with H.

We have found a second supersymmetry which anticommutes with the original one; its closure does not give a new operator independent of H. However, in view of the results (81) and (83), it is not appropriate to say that our theory has $N = 4$ supersymmetry. We have simply found two additional supercharges naturally associated with the structure constants of $\mathcal{G}$, a consideration that is built on significantly in the next subsection.

6.2 Hidden supersymmetries

One obvious question asks whether it is possible to find new supercharges that generalize those of Sec. 6.1. Consider the case of charges constructed using the five-cocycle $\Omega_{i_1 \ldots i_5}$ rather than the three-cocycle C_{ijk}. An analysis of the possibilities, q_{50}, q_{41}, q_{32}, q_{23}, q_{14}, q_{05}, leads one to conclude that only

$$q_{05} = \tfrac{1}{5!} i \Omega_{i_1 \ldots i_5} \pi_{i_1} \ldots \pi_{i_5} \ , \quad q_{50} = \tfrac{1}{5!} i \Omega_{i_1 \ldots i_5} c_{i_1} \ldots c_{i_5} \tag{87}$$

are hidden conserved supercharges because only they anticommute with q_{21} and q_{12}.

Moreover, we have the following general result:

The $l = \operatorname{rank} \mathcal{G}$ pairs of fermionic charges

$$\begin{aligned} q_{0,2m-1} &= \frac{i}{(2m-1)!} \Omega_{i_1 \ldots i_{2m-1}} \pi_{i_1} \ldots \pi_{i_{2m-1}} \ , \\ q_{2m-1,0} &= \frac{i}{(2m-1)!} \Omega_{i_1 \ldots i_{2m-1}} c_{i_1} \ldots c_{i_{2m-1}} \ , \end{aligned} \tag{88}$$

determined by the $(2m-1)$*-cocycles of the Lie algebra* $\mathcal{G}$*, where the allowed values of* m *depend on the specific* $\mathcal{G}$ *considered, also commute with* q_{21} *and* q_{12}*, and hence they commute with* H_q *and are conserved supercharges.*

Proof: Let us restrict ourselves to $q_{2m-1,0}$ (the case $q_{0,2m-1}$ is completely analogous). Consider first

$$\{q_{2m-1,0}, q_{12}\} \propto \Omega_{i[i_2\ldots i_{2m-1}} C_{j_1 j_2]i} c_{j_1} c_{j_2} c_{i_2} \ldots c_{i_{2m-1}} \ , \tag{89}$$

where the antisymmetrization is forced by the presence of the c's. This expression vanishes by the G-invariance of Ω, since this implies

$$\Omega_{i[i_1\ldots i_{2m-2}} C_{i_{2m-1}]ij} = 0 \ . \tag{90}$$

Hence, $\{q_{2m-1,0}, q_{12}\} = 0$. Now we have to check that the following anticommutator also vanishes:

$$\begin{aligned}\{q_{2m-1,0}, q_{21}\} \ \propto \ & \Omega_{ii_2\ldots i_{2m-1}} C_{ij_1j_2} c_{j_1} c_{i_2} \ldots c_{i_{2m-1}} \pi_{j_2} \\ & +(m-1)\Omega_{iji_3\ldots i_{2m-1}} C_{ijj_1} c_{j_1} c_{i_3} \ldots c_{i_{2m-1}} \ .\end{aligned} \tag{91}$$

The first term vanishes due to the antisymmetry in the indices $j_1, i_2, \ldots, i_{2m-1}$ and the invariance of Ω. To show that the second term also vanishes, we have to prove that

$$D \equiv \Omega_{iji_1\ldots i_{2m-3}} C_{i_{2m-2}ij} c_{i_1} \ldots c_{i_{2m-2}} \tag{92}$$

is equal to zero. Indeed, the invariance of Ω allows us to write it in terms of the C's and the invariant symmetric tensor $t_{l_1\ldots l_m}$ (see (64)) without having to involve i, j in the antisymmetrization, so we have

$$D = C_{ii_1l_1} C_{i_2i_3l_2} \ldots C_{i_{2m-4}i_{2m-3}l_{m-1}} t_{l_1\ldots l_{m-1}j} C_{i_{2m-2}ij} c_{i_1} \ldots c_{i_{2m-2}} \ . \tag{93}$$

Now, using the Jacobi identity

$$C_{ii_1l_1} C_{i_{2m-2}ij} = C_{ii_{2m-2}l_1} C_{i_1ij} + C_{ii_1i_{2m-2}} C_{l_1ij} \quad , \tag{94}$$

we arrive at

$$\begin{aligned}D \ = \ & C_{ii_{2m-2}l_1} C_{i_2i_3l_2} \ldots C_{i_{2m-4}i_{2m-3}l_{m-1}} t_{l_1\ldots l_{m-1}j} C_{i_1ij} c_{i_1} \ldots c_{i_{2m-2}} \\ & +C_{ii_1i_{2m-2}} C_{i_2i_3l_2} \ldots C_{i_{2m-4}i_{2m-3}l_{m-1}} t_{l_1\ldots l_{m-1}j} C_{l_1ij} c_{i_1} \ldots c_{i_{2m-2}} \ .\end{aligned} \tag{95}$$

The first term of this expression is equal to $-D$ due to the presence of c_{i_1} and $c_{i_{2m-2}}$, and the second term vanishes because C_{l_1ij} is antisymmetric in

l_1, j whereas $t_{l_1 \dots l_m - 1j}$ is symmetric in these indices. So we have $D = -D$, $D = 0$, $\{q_{2m-1,0}\, , q_{21}\} = 0$ and $[q_{2m-1,0}\, , H_q] = 0$, *q.e.d.*.

Hence, the following result follows:

For every simple Lie algebra $\mathcal{G}$ of rank l, the model (72) has a series of $2l$ conserved supercharges that are constructed from the primitive cocycles of $\mathcal{G}$, which include the supersymmetry generators.

7 Hidden supersymmetries as Noether charges

All the hidden supercharges discussed can be shown to be Noether charges associated with actual supersymmetries of the actions of the models in question. One way to realise this is to use the quantum commutator of the supercharge and the variables of the model to extract the variations. Explicitly, if $\tilde{Q}$ is the conserved supercharge and u is a generic component field in the model, its variation may be defined by

$$\delta_{\tilde{\epsilon}} u = [\tilde{\epsilon}\tilde{Q}, u] \ , \tag{96}$$

where $\tilde{\epsilon}$ is the corresponding fermionic parameter.

If $\tilde{Q}$ is a symmetry of the classical action $S = \int dt L$, $\delta_{\tilde{\epsilon}} S = 0$ for the constant parameter $\tilde{\epsilon}$. This means that, if we allow $\tilde{\epsilon}$ to become a function of t, its variation will be of the form

$$\delta_{\tilde{\epsilon}} S = \int dt(-i\dot{\tilde{\epsilon}}\tilde{Q}) \tag{97}$$

ignoring boundary terms, where $\tilde{Q}$ is the conserved Noether charge for the symmetry (96). Indeed, from (97) we get $\delta_{\tilde{\epsilon}} S = \int dt(i\tilde{\epsilon}\dot{\tilde{Q}})$, and since for solutions of the Lagrange equations $\delta_{\tilde{\epsilon}} S$ must be zero for any δ, it follows that $\dot{\tilde{Q}} = 0$ and $\tilde{Q}$ is the conserved charge. This procedure is particularly suitable when, as here, the complications addressed in [18] do not arise.

We now give the variations obtained by using (96). In the bosonic case with $A_i \neq 0$, use of (10) yields

$$\delta_{\tilde{\epsilon}} x_i = -i\tilde{\epsilon}_i f_{ijk} x_k \psi_j \ , \quad \delta_{\tilde{\epsilon}} \psi_i = \tilde{\epsilon} \dot{x}_j f_{jik} x_k - i\tilde{\epsilon} f_{jki} \psi_j \psi_k \tag{98}$$

for the variation induced by $\tilde{Q}_3$ (eq. (25)). If now we put $\epsilon = \epsilon(t)$ and ignore boundary terms in the integrand, we do find $\delta L = -i\dot{\epsilon}\tilde{Q}_3$, recovering $\tilde{Q}_3$ as the Noether charge. The same applies to the other supercharges below.

Consider $\tilde{Q}_5$ (eq. (31)). The corresponding formulae for the variations for the $A=0$ model of Sec. 3.2 are,

$$\delta_{\tilde{\epsilon}} x_i = -i\tilde{\epsilon}\Omega_{ijkpq} x_q \psi_{jkp} \ , \quad \delta_{\tilde{\epsilon}}\psi = 3\tilde{\epsilon}\dot{x}_j \Omega_{jkpiq} x_q \psi_{kp} - i\tilde{\epsilon}\Omega_{jkpqi}\psi_{jkpq} \ . \qquad (99)$$

Note that in the above variations $\delta\psi_i$ involves the derivative of $\dot{x}$ (mathematically, this means that succesive tangent spaces – jet spaces – are needed to define the action of the $\tilde{Q}$'s). This is not the case for the two purely fermionic models for which the canonical quantum commutators give

$$\delta_{\tilde{\epsilon}}\psi_i = (2m-1)\tilde{\epsilon}\Omega_{i_1\ldots i_{2m-2}i}\psi_{i_1\ldots i_{2m-2}} \qquad (100)$$

for $\tilde{Q}_{2m-1}$ (eq. (65)), for the $N=1$ model of Sec. 5.1, and

$$\begin{aligned} \delta_{\tilde{\epsilon}^*}\mu_i &= \frac{i\tilde{\epsilon}^*}{(2m-2)!}\Omega_{i_1\ldots i_{2m-2}i}\mu^*_{i_1}\ldots\mu^*_{i_{2m-2}} \ , \quad \delta_{\tilde{\epsilon}^*}\mu^*_i = 0 \ , \\ \delta_{\tilde{\epsilon}}\mu^*_i &= \frac{i\tilde{\epsilon}}{(2m-2)!}\Omega_{i_1\ldots i_{2m-2}i}\mu_{i_1}\ldots\mu_{i_{2m-2}} \ , \quad \delta_{\tilde{\epsilon}}\mu_i = 0 \end{aligned} \qquad (101)$$

for the variations produced, respectively, by the supercharges $q_{0,2m-1}$ and $q_{2m-1,0}$ (eqs. (88)), of the N=2 model of Sec. 6.1.

To proceed further, consider first the closure of δ_ϵ, $\delta_{\tilde{\epsilon}}$ on x_i, say, for the $A \neq 0$ model in Sec. 4. First we find

$$[\delta_\epsilon, \delta_{\tilde{\epsilon}}]x_i = 0 \qquad (102)$$

reflecting the fact that $\{Q, \tilde{Q}_3\} = 0$ (eq. (44)). For $[\delta_{\tilde{\epsilon}'}, \delta_{\tilde{\epsilon}}]$ we find

$$[\delta_{\tilde{\epsilon}'}, \delta_{\tilde{\epsilon}}]x_i = -2i\tilde{\epsilon}\tilde{\epsilon}' f_{ijk} J_j x_k \ , \qquad (103)$$

where

$$J_i = f_{ijk} x_j \dot{x}_k - \tfrac{1}{2} i f_{ijk}\psi_j\psi_k \qquad (104)$$

is the conserved charge (*cf.* (29)) associated with the adjoint transformations $\delta x_i = f_{ijk} a_j x_k$, $\delta\psi_i = f_{ijk} a_j \psi_k$ which leave the Lagrangian (8) invariant. This of course agrees with the variations on x_i induced by the operator $\{\tilde{Q}_3, \tilde{Q}_3\}$ expressed in the form (*cf.* again (29)) $\vec{J}^2 + \frac{1}{3}\vec{S}^2$. A similar analysis can be performed for the corresponding actions upon ψ_i, for which we get

$$[\delta_{\tilde{\epsilon}'}, \delta_{\tilde{\epsilon}}]\psi_i = -2i\tilde{\epsilon}\tilde{\epsilon}' f_{ijk} J_j \psi_k - 2i\tilde{\epsilon}\tilde{\epsilon}' f_{jlm} f_{jik} x_m x_k \dot{\psi}_l \ , \qquad (105)$$

in which we see the second term is zero on shell, using $\dot{\psi}_l = qF_{lj}\psi_j$ where F_{lj} is given by (47).

8 Concluding remarks

We have shown in this paper that there exist 'hidden' supersymmetric fermionic charges associated with the Lie algebra cohomology cocycles of the symmetry group for three simple $Ad(G)$-invariant supersymmetric models, one with bosonic and fermionic coordinates and two (for $N=1$ and $N=2$) with only fermionic dynamical coordinates. For the first one, and in the free case, there are l additional supersymmetries the existence of which is tied to the Killing-Yano tensors of valence $(2m-2)$ that may be constructed from the $(2m-1)$ Lie algebra cohomology cocycles. In the interacting $A \neq 0$ case the same procedure seems to allow for only one additional supersymmetry, associated with the structure constants f_{ijk} of the Lie algebra $\mathcal{G}$ considered. In the $N=1$ fermionic model, at least in the case of $\mathcal{G} = su(n)$ l additional symmetries may be constructed directly from its cocycles. In general, these purely fermionic $\tilde{Q}$'s depend only on the cohomology of $\mathcal{G}$ or, equivalently, on the topology of the corresponding compact group G. In this sense the additional supercharges may be traced to the topology of G; however, they may be seen to generate continuous symmetries of the system and may be obtained from Noether's theorem. In the $N=2$ case it is shown that the standard supercharges are in fact part of a series of $2l$ conserved supercharges that can be constructed from the l cocycles of $\mathcal{G}$.

Summarising, our analysis shows that the hidden symmetries appearing in $Ad(G)$-invariant models are in fact supersymmetries because they stem from the existence of the G-invariant odd skewsymmetric tensors associated with the cohomology of $\mathcal{G}$. In this sense, these additional fermionic charges could have been introduced directly, even before having a supersymmetric model, as in the ($\mathcal{G} = su(2)$) analysis of [19] of the results of [8]. However, the fact that they square to a Casimir has more to do with the structure of the cocycles themselves [10], and hence with the structure of the generic symmetry group, than with any other considerations of $su(2)$. In fact, the expression (29) (see also (56)), that holds for *any* Lie algebra $\mathcal{G}$ explains why *e.g.* $\{\tilde{Q}_3, \tilde{Q}_3\}$ is given by a Casimir. Also, our analysis in sec. 4 shows that the rotational symmetry of the model in [8] does not play an essential role (cf. [19]), being just a result of the fact that the fully antisymmetric tensor in 3-dimensions provides the structure constants.

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